

Mass gap, Abelian Dominance and Vortex Dynamics in $SU(2)$ Spin Model

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ABSTRACT

We discuss a new approach to the investigation of the nature of the mass gap in spin systems with continuous global symmetries which is much analogous to the method of abelian projection in the gauge theories. We suggest that the abelian degrees of freedom, in particular, abelian vortices are responsible for the mass gap generation phenomena in the non-abelian spin systems. To check our hypothesis we study numerically the three-dimensional $SU(2)$ spin model in the Maximal Abelian projection. We find that the abelian mass gap in the projected theory coincides with the full non-abelian mass gap within numerical errors. The study of the percolation properties of the abelian vortex trajectories shows that the phase transition and the mass gap generation in the 3D $SU(2)$ spin model are driven by the abelian vortex condensation.

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1 Introduction

This paper proposes a new approach to the investigation of the mass gap nature in the spin systems with nonabelian symmetry group. While this question is well studied in the context of $O(2)$ spin system, it is open for non-abelian spin systems. The situation much resembles the problem with the string tension in pure gauge theory where the monopoles are known to be responsible for the confinement in the strong coupling lattice $U(1)$ theory [1] but the nature of the nonzero string tension in a nonabelian theory at arbitrary couplings is still unclear. The main question of all studies devoted to these problems is what are relevant configurations which lead to nonzero string tension in gauge theories or mass gap in spin systems, correspondingly.

During last decade there was some progress in understanding the confining forces in the $SU(N)$ gauge theories. In particular, this progress was done in the framework of partially fixed gauges, for example, within abelian projection approach [2]. This approach is based on the partial gauge fixing of the $SU(N)$ gauge degrees of freedom to an “abelian gauge”, such that the maximal torus group $[U(1)]^{N-1}$ remains unfixed. After the partial gauge fixing, abelian monopoles appear due to compactness of the residual abelian gauge group. According to the dual superconductor picture [3], the condensation of such monopoles may explain the colour confinement in gluodynamics: if the monopoles are condensed then the confining string is formed between the quark-antiquark pair due to the dual Meissner effect. This picture has been confirmed in various numerical simulations of lattice gluodynamics (see, for example, the reviews [4, 5]).

The mass gap has also been intensively investigated in the context of spin models. Berezinskii, Kosterlitz and Thouless [6] showed that the condensation of vortices in $2D$ XY model leads to the spin-spin correlations which fall down exponentially as distance between spins increases. Thus the vortex condensation gives rise to the nonzero mass gap in the relevant region of the coupling constant [6]. Up to now it is not much known, however about the nature of the mass gap in nonabelian spin systems (see, [7] and references therein on the general review of the subject). The usual methods for the studying of the mass gap in the spin models (high and low temperature expansions, Monte-Carlo simulations) do not deal much with the nature of the mass gap. These methods rather concentrate on other problems like behaviour of the mass gap in the finite volume and in the perturbation theory, comparison of the mass gap which is calculated in Monte-Carlo simulations with the Bethe ansatz prediction, *etc.* (see, for instance, [8]). Even “exact” mass gap (calculated in $2D$ $O(N)$ models under assumptions of validity of the Bethe ansatz and the standard perturbation theory) tells us exactly nothing about relevant configurations of spins contributing to the mass gap [9].

Meantime, it would be very desirable and suggestive to have a deeper insight into the nature and properties of the mass gap in nonabelian spin models. We are interested in a possible generalization of the vortex mechanism of the mass gap generation in the XY model to the nonabelian spin systems. One can also elaborate mechanisms of the mass gap generation, which are analogous to the confinement mechanisms in gauge theories. It seems to be natural then to apply known methods of the investigation of abelian degrees of freedom in gauge systems to the spin systems. In this paper we

propose to extend the method of abelian gauge fixing to the spin systems in order to find spin configurations which are relevant for the mass gap generation. The Maximal Abelian (MaA) projection [10] is of a special interest for us in this respect since this projection is a good framework to display the dynamics of abelian degrees of freedom of the gauge theory.

In Section 2 we define the Maximal Abelian projection for $SU(2)$ spin model and we show that after the abelian projection the $SU(2)$ spin model becomes $[O(2)]^2$ spin model with non-local interaction between spins. This abelian system possesses two types of the $O(2)$ vortices. We argue that these vortices as well as the corresponding abelian spins may have different relevance to the dynamics of the $SU(2)$ spin system in the MaA projection. We also stress the difference between projections in gauge and spin models. The present publication is devoted to the study of most relevant type of the abelian degrees of freedom. More detailed investigation of the properties of abelian system as well as the discussion of the $SU(N)$ spin models will be presented elsewhere [15].

In Section 3 we study numerically the abelian mass gap which is obtained from abelian spin-spin correlators in the MaA projection and compare it with the $SU(2)$ mass gap. We find that $SU(2)$ mass gap coincides with its abelian counterpart within numerical errors. This result is similar to the effect of “abelian dominance” in lattice gluodynamics [11, 14]. The investigation of the vortex percolation properties shows that the abelian vortices are condensed in the massive phase and they are dilute in the massless phase.

We conclude that i) the abelian degrees of freedom in the MaA gauge are relevant degrees of freedom for the mass gap generation in $SU(2)$ spin model; ii) the mechanisms of the mass gap generation in $3D$ $SU(2)$ spin system in the MaA projection is similar to that in $3D$ XY model [16, 17, 18]: the abelian vortex condensation leads to the mass gap generation.

2 Maximal Abelian Projection for $SU(2)$ Spins

We consider the $SU(2)$ spin model with the action

$$S = -\frac{\beta}{2} \sum_x \sum_{\mu=1}^D \text{Tr} U_x U_{x+\mu}^+, \quad (1)$$

where U_x are the $SU(2)$ spin fields, $\beta = \frac{2}{g^2}$ and D is the dimension. The action (1) is invariant under $SU_L(2) \times SU_R(2)$ global transformations:

$$U_x \rightarrow U_x^{(\Omega)} = \Omega_L^+ U_x \Omega_R, \quad \Omega_{L,R}^+ \in SU(2). \quad (2)$$

In analogy with the abelian projection in the non-abelian gauge theories [2], we define the abelian projection in the non-abelian spin systems as a gauge fixing of the non-abelian symmetry group up to its maximal torus subgroup.

Let us recall the definition of the MaA projection in $SU(2)$ lattice gluodynamics [10]. The gauge fixing condition is defined as the maximization of the functional

$$R_g[\mathcal{U}] = \sum_{x,\mu} \text{Tr}(\mathcal{U}_{x,\mu} \sigma^3 \mathcal{U}_{x,\mu}^+ \sigma^3), \quad (3)$$

over all $SU(2)$ gauge transformations $\mathcal{U}_{x,\mu} \rightarrow \Omega_x^+ \mathcal{U}_{x,\mu} \Omega_{x+\hat{\mu}}$,

$$\max_{\{\Omega\}} R_{gauge}[\mathcal{U}^{(\Omega)}], \quad (4)$$

here $\mathcal{U}_{x,\mu}$ are the $SU(2)$ gauge link fields and σ^a are the Pauli matrices.

The functional R_g is invariant under $U(1)$ gauge transformations $\tilde{\Omega}_x = e^{i\sigma_3 \alpha_x}$, $\alpha \in [0, 2\pi)$, and therefore it fixes the $SU(2)$ gauge group up to the diagonal $U(1)$ subgroup. The residual abelian group is compact, and the abelian projected theory contains the abelian monopoles. The numerical simulations of the $SU(2)$ gluodynamics in the Maximal Abelian projection shows that the abelian degrees of freedom, in particular the abelian monopoles are responsible for the confinement phenomenon [4, 5]. There are a lot of the abelian projections of the $SU(2)$ gluodynamics but the abelian degrees of freedom in the MaA projection seem to be the most relevant to infrared properties of this gauge theory. Therefore we may expect that the abelian degrees of freedom in the spin system in the MaA projection are responsible for the infrared structure of the spin model.

We define the Maximal Abelian projection for the $SU(2)$ spin theory by the following maximization condition:

$$\max_{\{\Omega\}} R_s[U^{(\Omega)}], \quad (5)$$

where the maximized functional is the analog of functional (3):

$$R_s[U] = \sum_x \text{Tr}(U_x \sigma^3 U_x^+ \sigma^3), \quad (6)$$

The functional $R_s[U]$ is invariant under $U_L(1) \times U_R(1)$ global transformations:

$$U_x \rightarrow U_x^{(\tilde{\Omega})} = \tilde{\Omega}_L^+ U_x \tilde{\Omega}_R; \quad \tilde{\Omega}_{L,R} = e^{i\sigma^3 \omega_{L,R}}, \quad \omega_{L,R} \in [0, 2\pi). \quad (7)$$

Due to the invariance (7), the condition (5) fixes the $SU_L(2) \times SU_R(2)$ global symmetry group up to $U_L(1) \times U_R(1) \sim O_L(2) \times O_R(2)$ global group.

The important difference between the abelian gauge fixing for the spin and for the gauge systems is that in the spin system we fix the *global* symmetry. Thus the number of the gauge conditions does not depend on the lattice volume while the symmetry group in the gauge system is local and the number of the gauge conditions grows

proportionally to the volume. Therefore the projection in gauge system fixes each link matrix to be as close to diagonal as possible, while the projection in the spin system forces an average value of the spin to be diagonal. Therefore, the projection for the spin systems is somehow simpler to handle than for gauge theories.

Let us parametrize the $SU(2)$ spin field U in the standard way: $U_x^{11} = \cos \varphi_x e^{i\theta_x}$; $U_x^{12} = \sin \varphi_x e^{i\chi_x}$; $U_x^{22} = U_x^{11,*}$; $U_x^{21} = -U_x^{12,*}$; $0 \leq \varphi \leq \pi/2$, $0 < \theta, \chi \leq 2\pi$. Under the $O_L(2) \times O_R(2)$ transformation (7), components of the field U transform as

$$\theta_x \rightarrow \theta'_x = \theta_x + \omega_d \mod 2\pi, \quad \chi_x \rightarrow \chi'_x = \chi_x + \omega_o \mod 2\pi, \quad (8)$$

where $\omega_{d,o} = -\omega_L \mp \omega_R$. It is convenient to decompose the residual symmetry group, $O_L(2) \times O_R(2) \sim O_d(2) \times O_o(2)$. The diagonal (off-diagonal) component θ (χ) of the $SU(2)$ spin U transforms as a spin variable with respect to the $O_d(2)$ ($O_o(2)$) symmetry group.

It is useful to consider the one-link spin action S_l in terms of the angles φ , θ , χ :

$$S_{l_{x,\mu}} = -\beta \left[\cos \varphi_x \cos \varphi_{x+\hat{\mu}} \cos(\theta_x - \theta_{x+\hat{\mu}}) + \sin \varphi_x \sin \varphi_{x+\hat{\mu}} \cos(\chi_x - \chi_{x+\hat{\mu}}) \right]. \quad (9)$$

The action consists of two parts which correspond to the self-interaction of the spins θ and χ , respectively. The $SU(2)$ component φ does not behave as a spin field and its role is to provide the interaction between the θ and χ spins. Thus, the $SU(2)$ spin model in the abelian projection reduces to two interacting copies of the XY model with the fluctuating couplings due to the dynamics of the field φ .

Two types of the abelian vortices appear in the abelian projected $SU(2)$ spin model due to the compactness of the residual abelian group. They correspond to the diagonal (θ) and to the off-diagonal (χ) abelian spins. We call these vortices as “diagonal” and “off-diagonal” vortices, respectively.

We expect that in the MaA projection the diagonal vortices may be more dynamically important than the off-diagonal vortices. The reason for this expectation is simple. In our representation for $SU(2)$ spin field the maximizing functional (6) has the form: $R_{spin} = 4 \sum_x \cos^2 \varphi_x + const$. Therefore, in the MaA projection the effective coupling constant in front of the action for the θ spins is maximized while the effective self-coupling for χ spins is minimized. Thus we may expect that the diagonal vortices may be more relevant to the dynamics of the system than the off-diagonal vortices^a.

One should note that both types of the abelian vortices as well as any other $[O(2)]^2$ invariant quantities in the MaA projection can be easily defined in the $SU(2)$ invariant way [15]. Therefore the abelian observables which are discussed in our work cannot be gauge artifacts.

^aThere exists similar situation in the lattice gluodynamics in the MaA projection: the action for the diagonal variables of the $SU(2)$ gauge fields is larger than the action for the non-diagonal variables [12]. The numerical simulations show [12] that topological defects in the diagonal fields, monopoles, are related to the dynamics of the gauge theory, while off-diagonal defects, minipoles, decouple from the dynamics of the gauge system.

3 Results of Numerical Simulations

In the present Section we study the abelian mass gap for the diagonal spins and the behaviour of the diagonal vortices in the three dimensional $SU(2)$ spin model on the lattices $L^2 \times L_z$, $L = 16$, $L_z = 16, 32$ with periodic boundary conditions.

We calculate the non-abelian mass gap from the plane-plane correlator:

$$\langle \text{Tr}(\tilde{U}(0) \cdot \tilde{U}^+(z)) \rangle = \text{const.} \left(e^{-\mu_{SU(2)} \cdot z} + e^{-\mu_{SU(2)} \cdot (L_z - z)} \right) + \dots, \quad (10)$$

where $\tilde{U}(z) = L^{-1} \sum_{x,y=1}^L U(x, y, z)$ is the zero-momentum spin operator in the $x - y$ plane and terms vanishing in the limit $L_z \rightarrow \infty$ are denoted by dots. Mass gap is calculated by fitting the zero-momentum spin-spin correlator by the leading term in eq.(10).

The abelian mass gap can be calculated similarly. We use the zero-momentum *abelian* spin operator $\tilde{U}_{ab}(z) = L^{-1} \sum_{x,y=1}^L \exp\{i \theta(x, y, z)\}$ and measure the correlator:

$$\langle \tilde{U}_{ab}(0) \cdot \tilde{U}_{ab}^*(z) \rangle_{MaA} = \text{const.} \left(e^{-\mu_{O(2)} \cdot z} + e^{-\mu_{O(2)} \cdot (L_z - z)} \right) + \dots, \quad (11)$$

where the quantum average is taken in the MaA gauge (5,6). Zero momentum correlator (11) can be defined [15] as a quantum average of some non-local $SU(2)$ -invariant operator; this average can be calculated without gauge-fixing. This shows that in a general gauge the zero-momentum operator which corresponds to the abelian spin-anti-spin in the MaA gauge and the zero-momentum non-abelian spin-anti-spin operator are related to each other in a non-local way.

In numerical simulations we use U.Wolff's cluster algorithm [21], which is known to work perfectly well in the $O(N)$ spin models. In particular, this algorithm has no problems with critical slowing down. In order to thermalize our fields at each β value we performed a number of thermalization sweeps, which is much greater than measured auto-correlation time.

The abelian and the $SU(2)$ mass gaps *vs.* β are shown on Fig. 1. The plotted mass gap is very small in the massless phase ($\beta > \beta_c = 0.935 \dots$) but it is non zero due to the finite-volume effects and fitting technique. The abelian mass gap coincides with the $SU(2)$ mass gap within numerical errors. This result is similar to the already known numerical result in the lattice gluodynamics: the abelian string tension (derived from abelian Wilson loops) almost coincides with the full $SU(2)$ string tension [11, 13]. This effect was first numerically observed in gluodynamics in Ref. [11], and it is called the “abelian dominance” [11, 14]. In the present paper we find the abelian dominance in the $SU(2)$ spin system: the non-abelian mass gap generation is dominated by the abelian degrees of freedom.

Due to the abelian dominance the abelian topological defects (abelian vortices) may play a role in the mass gap generation. The vortex trajectory $*j$ is constructed from the spin variables θ by the formula [1, 19]: $*j = (2\pi)^{-1} *d[\theta]$, where the square brackets stand for “modulo 2π ” and the operator “d” is the lattice derivative^b. In the

^bWe use the formalism of the differential forms on the lattice [20].

three dimensions the vortex trajectories are loops which are closed due to the property $d^2 = 0$.

In order to have a feeling on the behaviour of the vortices we visualized the vortex trajectories in the different phases of the spin system. The examples of the typical diagonal vortex configurations in the massive ($\beta = 0.8$) and the massless ($\beta = 1.1$) phases are shown on Fig. 2(a) and Fig. 2(b), respectively.

It is clearly seen from Figs. 2(a, b) that the diagonal vortices form large clusters in the massive phase and they are dilute in the massless phase. The similar properties of the vortex loops have been found [17, 18] in the $3D$ XY model where the mass gap generation is known to be due to the vortex condensation [16, 17, 18].

One of the quantitative characteristics of the vortex ensemble is the so-called percolation probability^c [17] C :

$$C = \lim_{r \rightarrow \infty} \left(\sum_{x,y,i} \delta_{x \in *j_i} \delta_{y \in *j_i} \cdot \delta(|x - y| - r) \right) \cdot \left(\sum_{x,y} \delta(|x - y| - r) \right)^{-1}, \quad (12)$$

where the summation is going over all the vortex trajectories j_i , and over all the points x, y of the lattice. The quantity C has a meaning of the probability for two infinitely separated points of the lattice to be connected by a vortex trajectory. If this probability is not zero then the infinite long vortex loops exit in the vacuum of the spin system. The last fact implies the dominance of the entropy of the vortex trajectories over the vortex energy which means the existence of the vortex condensate. Therefore, if the vortices are condensed this probability is non-zero, and if the vortex condensate does not exist the quantity C must be zero [17].

The percolation probability C for the diagonal vortex trajectories is shown on Fig. 3. The percolation probability is not zero in the massive phase, $\beta < \beta_c$, thus in this phase the vortices are condensed. After the phase transition the percolation probability is getting smaller and it vanishes right after the phase transition point β_c . Therefore in the massless phase the vortices are not condensed. Our result shows that the phase transition in the $SU(2)$ spin model is driven by the abelian vortex condensation.

Conclusions and Acknowledgments

Our results show that the mass gap generation in $3D$ $SU(2)$ spin system can be described through the abelian degrees of freedom in the Maximal Abelian gauge. We found the abelian dominance for the mass gap generation: the non-abelian mass gap coincides within numerical errors with the abelian mass gap. The mechanism in the non-abelian spin model is similar to that in $3D$ XY model: the abelian vortex condensation leads to the mass gap generation.

In our next publication [15] we are going to investigate in detail the properties of both types of vortices and to develop the projection method for $2D$ $SU(N)$ spin systems.

^cThis formula is written for the infinite volume lattice. The finite volume definition can be found in Refs. [17, 18].

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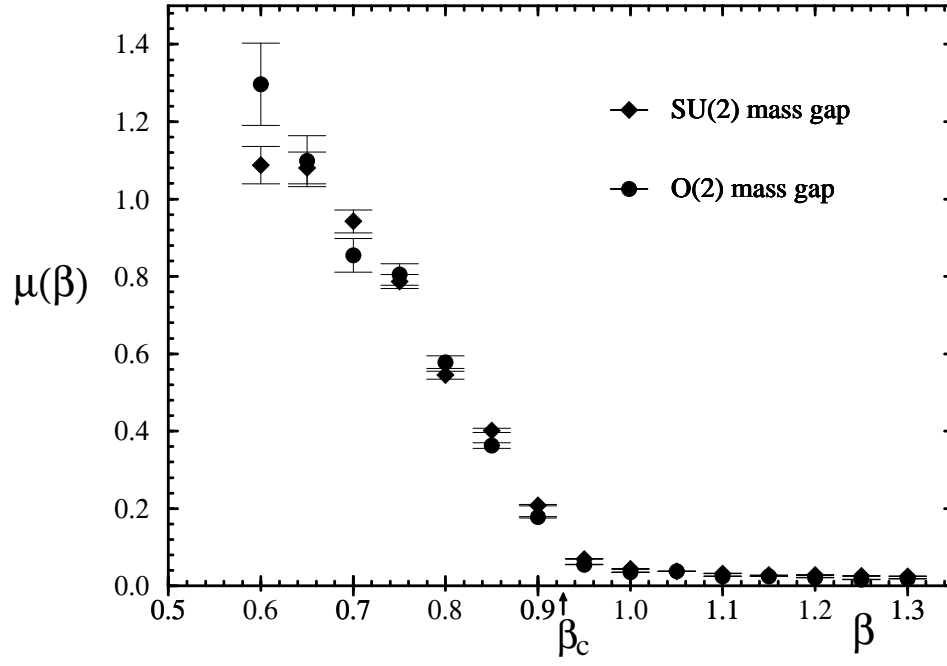
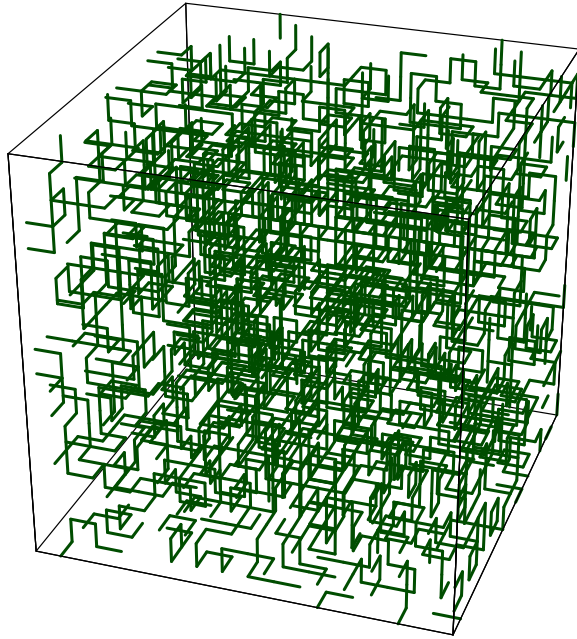
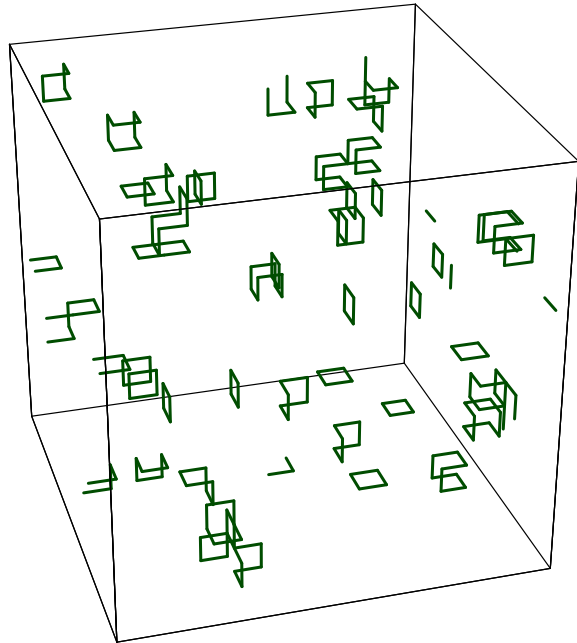


Fig. 1: The abelian and $SU(2)$ mass gaps *vs.* β , the lattice is $16^2 \cdot 32$.



(a)



(b)

Fig. 2: The examples of typical abelian diagonal vortex configurations in the massive (a) and the massless (b) phases on the 16^3 lattice.

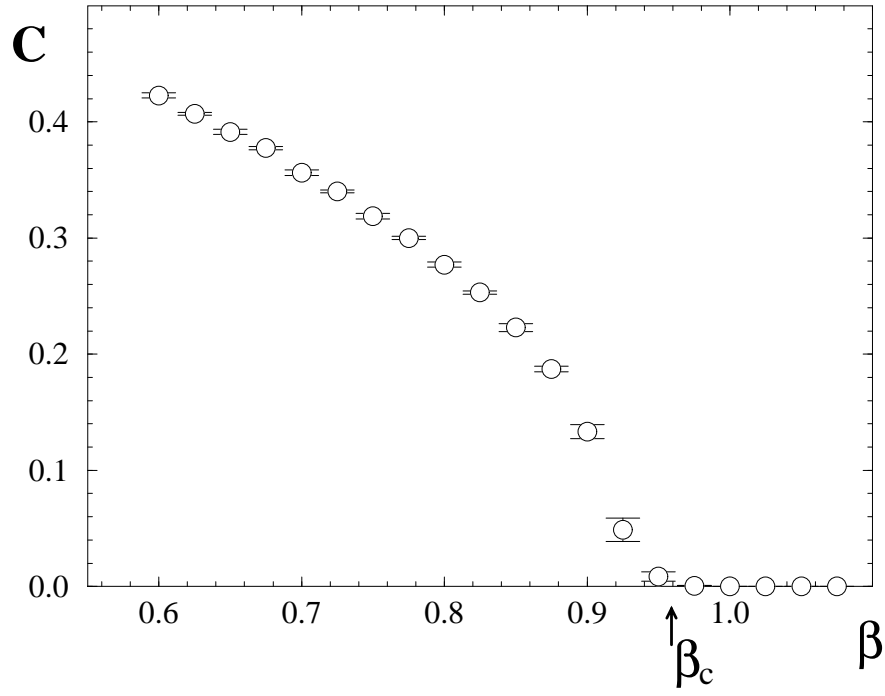


Fig. 3: The percolation probability C for the diagonal vortex trajectories *vs.* β on the 16^3 lattice.